

FORM-INVARIANCE UPON RELATIVISATION OF THE HAMILTONIAN STRUCTURES OF FLUIDS IS NOT UNIVERSAL

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Abstract—The principle of form-invariance upon relativisation of the Hamiltonian structures of fluids and plasmas is an empirical observation that the (special) relativistic version of a given non-relativistic Hamiltonian system of classical Physics (a) is also a Hamiltonian system, and moreover, (b) has the same noncanonical Hamiltonian structure as its nonrelativistic counterpart. It is shown that one-dimensional gas dynamics of a polytropic gas violates this principle.

INTRODUCTION

Continuous dynamics of nonrelativistic conservative systems is often governed by the noncanonical Hamiltonian formalism (see, e.g., bibliography in [1,2]). Experimental observations show that the (special) relativisation of such systems preserves their Hamiltonian character and even retains the nonrelativistic form of their Hamiltonian structures. This fact has been by now established for all the basic systems: adiabatic fluid dynamics in flat [3–5] and in curved [6, 1] space, nonabelian hydrodynamics [7], Yang–Mills–Vlasov Plasma [7], adiabatic Magnetohydrodynamics (MHD) [8], and anisotropic MHD [9]. This Hamiltonian rigidity with respect to special relativisation has been found to apply not only to individual dynamical systems but also to nontrivial morphisms between these systems, such as the reduction with respect to the direct sum of orthogonal groups [10]. It appears that the principle of Hamiltonian form-invariance under relativisation is universally applicable, but this is not to be. Below I show that relativisation of one-dimensional dynamics of arbitrary nonrelativistic polytropic gas destroys its second Hamiltonian structure.

NONRELATIVISTIC ONE-DIMENSIONAL GAS DYNAMICS

The motion equations of nonrelativistic one-dimensional barotropic fluid (or gas) dynamics are

$$-M_t = (Mv + P)_x \quad (1.1)$$

$$-\rho_t = (\rho v)_x \quad (1.2)$$

$$v = M/\rho \quad (2)$$

where M is the momentum density, ρ is the fluid density, v is the velocity, subscripts (except the subscript zero later on) denote partial derivatives, and $P = P(\rho)$ is the pressure,

$$P = \rho^2 e_p \quad (3)$$

where $e = e(\rho)$ is the specific internal energy. With the Hamiltonian being the energy density

$$H = H_2 = \frac{M^2}{2} + \rho e, \quad (4)$$

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the system (1), (2) has the standard Hamiltonian structure in the (M, ρ) -variables (with $\partial = \partial/\partial x$):

$$B^1 = \begin{pmatrix} M\partial + \frac{\partial M}{\partial \rho} & \rho\partial \\ \partial \rho & 0 \end{pmatrix} \quad (5)$$

which means two things: first, that the system (1), (2) can be written in the form

$$-\begin{pmatrix} M \\ \rho \end{pmatrix}_t = B^1 \begin{pmatrix} \delta H/\delta M \\ \delta H/\delta \rho \end{pmatrix} \quad (6)$$

where $\delta H/\delta(\cdot)$ is the variational (in our case, partial) derivative of H (4) with respect to the variable (\cdot) ; and second, that the matrix (5) is Hamiltonian (see, e.g., [12]).

The Hamiltonian representation (6) is available for *arbitrary* pressure function $P(\rho)$ or, what is the same, arbitrary specific internal energy function $e(\rho)$. In the case when the gas is polytropic,

$$P = \beta \rho^{1/\alpha}, \quad \beta \text{ and } 0 \neq \alpha \text{ are constants.} \quad (7)$$

Nutku observed [11] that there exists also another Hamiltonian structure of the dynamical system (1), (2):

$$B^2 = \begin{pmatrix} (Mv + P)\partial + \frac{\partial(Mv + P)}{\partial \rho v + \alpha M\partial} & \rho v\partial + \alpha \partial M \\ \partial \rho v + \alpha M\partial & \alpha(\rho\partial + \partial \rho) \end{pmatrix} \quad (8)$$

with the Hamiltonian function

$$H = H_1 = M. \quad (9)$$

It is this structure that will be shown to be impossible to deform with the parameter c^{-2} , c being the speed of light. Before proceeding further, a few remarks are in order.

REMARK 1. Nutku shows that for the polytropic gas there exists not only the second Hamiltonian structure B^2 but also a third Hamiltonian B^3 , with the Hamiltonian

$$H = H_0 = \rho \quad (10)$$

and that the third Hamiltonian structure services, in addition to polytropic gases (7), also the case

$$P = \beta \ell n \rho \quad (11)$$

REMARK 2. For the case $\alpha = 1/2$, the gas dynamical equations (1), (2) are isomorphic to the classical one-dimensional long-wave equations for which the three Hamiltonian structures were found earlier in [13] in the general context of deformations of Hamiltonian structures.

REMARK 3. For the case of a free fluid, i.e., when the pressure P vanishes [$\beta = 0$ in (7)], the multi-Hamiltonian representations were given in [14,15] (in more general relativistic and super cases).

REMARK 4. To check the Hamiltonian property of a given matrix B one has to check the following identity for arbitrary column-vectors X and Y (equation (2.17) in [15]):

$$B(\delta[Y^t B(X)]) = D[B(Y)][B(X)] - D[B(X)][B(Y)] \quad (12)$$

where $\delta(\cdot)$ is the column-vector of functional derivatives of (\cdot) and $D(\cdot)$ is the Fréchet derivative of (\cdot) . For the case

$$B = \begin{pmatrix} a\partial + \frac{\partial a}{\partial f} & f\partial + \frac{\partial g}{\partial b} \\ \partial f + g\partial & b\partial + \frac{\partial b}{\partial b} \end{pmatrix} \quad (13)$$

where a, b, f, g are arbitrary functions of u and h , the equation (12) reduces to the following system (equations VI (1.15) in [16]):

$$2bg_h + (f + g)g_u = 2ab_u + (f + g)b_h \quad (14.1)$$

$$2af_u + (f + g)f_h = 2ba_h + (f + g)a_u \quad (14.2)$$

$$g_h(a_u - f_h) = a_h(g_u - b_h) \quad (14.3)$$

$$b_u(a_u - f_h) = f_u(g_u - b_h) \quad (14.4)$$

$$a_h b_u = g_h f_u \quad (14.5)$$

Applying formulae (14) to the matrix (8) we find that B^2 is Hamiltonian if and *only if* the gas is polytropic:

$$\alpha \rho P_\rho = P. \quad (15)$$

Similarly one shows that a third Hamiltonian structure B^3 exists if and *only if* P is given either by (7) or (11).

REMARK 5. The function g and b (13) are taken in (8) to be

$$g = \alpha M, \quad b = \alpha \rho \quad (16)$$

under the assumption that the conserved density ρ/α of the dynamical equations (1), (2) is the momentum of the Hamiltonian matrix B^2 , i.e., it produces the uniform velocity-one shift in the x direction. A slightly more general choice would be to take

$$b = \alpha \rho + \mu, \quad \mu \text{ a constant}, \quad (17)$$

but one may check that no advantage is gained by allowing nonzero μ in (17).

REMARK 6. For the case (16), the system (14) collapses to the single equation (14.2), which takes the form

$$2af_M + (f + \alpha M)(f_\rho - a_M) = 2\alpha \rho a_\rho \quad (18)$$

Allowing v and P in (8) to be *arbitrary* functions of M and ρ , equation (18) becomes

$$Mv_M \left(v - \alpha \frac{M}{\rho} + \frac{2}{M} P \right) + \rho v_\rho \left(v - \alpha \frac{M}{\rho} \right) = P_M \left(v + \alpha \frac{M}{\rho} \right) + 2\alpha P_\rho, \quad (19)$$

which reduces to equation (15) when $v = M/\rho$ and $P_M = 0$.

REMARK 7. The matrices B^1 (5) and B^2 (8), (2), (7) are not only separately Hamiltonian but form a Hamiltonian pair, i.e., their arbitrary linear combination (with constant coefficients) is Hamiltonian as well. Once again allowing v and P in (8) to be arbitrary functions of M and ρ (anticipating relativistic complications later on), we find that the Hamiltonian matrices B^1 (5) and B^2 (8) form a Hamiltonian pair iff the equation (19) is satisfied for $v + \lambda$, for every constant λ , which in turn results in the equations

$$Mv_M + \rho v_\rho = P_M \quad (20)$$

$$\alpha (MP_M + \rho P_\rho) = \rho P v_M \quad (21)$$

Again, when $v = M/\rho$, one at once recovers equation (15).

RELATIVISTIC ONE-DIMENSIONAL GAS DYNAMICS

Relativistic gas dynamical equations have the same form (1) but with relations (2) and (3) being replaced by [4]

$$M = \theta v \quad (22)$$

$$\theta = \rho \gamma w \quad (23)$$

$$\gamma = (1 - v^2/c^2)^{-1/2} \quad (24)$$

$$w = 1 + (e_0 + P/\rho_0)c^{-2} \quad (25)$$

$$\rho = \rho_0 \gamma \quad (26)$$

$$P = P(\rho_0) = \rho_0^2 e_{\rho_0}, \quad e = e(\rho_0). \quad (27)$$

As shown in [4,5], the relativistic system (1), (22)–(27) possesses the same Hamiltonian form (5), with the Hamiltonian

$$H = \overline{H}_2 = c^2(\theta - \rho) - P \quad (28)$$

To show that second Hamiltonian form (8) does not survive relativisation we need to show that the defining relation (19) is not satisfied whatever $P(p_0)$ is provided $P \neq 0$. (When $P = 0$, one gets $w = 1$ and since, in general,

$$\theta = (\rho^2 w^2 + M^2/c^2)^{1/2}, \quad (29)$$

for the case $w = 1$ one obtains $\theta = \rho(1 + M^2/\rho^2 c^2)^{1/2}$. Thus,

$$v = \frac{M}{\rho} \left(1 + \frac{M^2}{\rho^2 c^2} \right)^{-1/2} \quad (30)$$

is of homogeneous degree zero in M and ρ . Therefore, since $P = 0$, both equations (20) and (21) are satisfied.) Now suppose $P \neq 0$. Since

$$M = \gamma^2 v \rho_0 w, \quad \rho = \gamma \rho_0, \quad (31)$$

we can re-express the partial derivatives operators ∂_M and ∂_ρ entering into the equation (19) in terms of ∂_v and ∂_{ρ_0} , as follows:

$$\partial_M = \Delta (R^1 \partial_{\rho_0} + R^2 \partial_v), \quad \partial_\rho = \Delta (Q^1 \partial_{\rho_0} + Q^2 \partial_v) \quad (32)$$

$$\Delta^{-1} = [\gamma^2 \gamma_v v \rho_0 (\rho_0 w)_{\rho_0} - \gamma (\gamma^2 v)_v \rho_0 w] \neq 0, \quad (33)$$

$$R^1 = \gamma_v \rho_0, \quad R^2 = -\gamma, \quad Q^1 = -(\gamma^2 v)_v \rho_0 w, \quad Q^2 = \gamma^2 v (\rho_0 w)_{\rho_0}. \quad (34)$$

Since equation (19) is linear in the (M, ρ) -partial derivatives, dividing out by Δ we can rewrite equation (19) in the form

$$MR^2 \left(v - \alpha \frac{M}{\rho} + \frac{2}{M} P \right) + \rho Q^2 \left(v - \alpha \frac{M}{\rho} \right) = P_{\rho_0} \left[R^1 \left(v + \alpha \frac{M}{\rho} \right) + 2\alpha Q^1 \right]. \quad (35)$$

Since $MR^2 + \rho Q^2 = \gamma^3 v \rho_0^2 w_{\rho_0}$, equation (35) becomes

$$\gamma^3 v^2 \rho_0^2 w_{\rho_0} (1 - \alpha \gamma w) - 2\gamma P = P_{\rho_0} [\gamma_v v \rho_0 (1 + \alpha \gamma w) - 2\alpha (\gamma^2 v)_v \rho_0 w]. \quad (36)$$

In the limit $c^{-2} \rightarrow 0$, one has $\gamma \rightarrow 1$, $\gamma_v \rightarrow 0$, $w \rightarrow 1$, $w_{\rho_0} \rightarrow 0$, and equation (36) yields $P = P_{\rho_0} \alpha \rho_0$, which is exactly equation (15). Using the formulae

$$w_{\rho_0} = P_{\rho_0} \rho_0^{-1} c^{-2} \quad (37)$$

and

$$P = P_{\rho_0} \alpha \rho_0 \quad (38)$$

in equation (36), we can reduce equation (36), by dividing it out by $P_{\rho_0} \rho_0$ ($P_{\rho_0} \rho_0 = \alpha^{-1} P \neq 0$), to the equation

$$\gamma^3 v^2 c^{-2} (1 - \alpha \gamma w) - 2 \gamma \alpha = \gamma_v v (1 + \alpha \gamma w) - 2 \alpha w (\gamma^2 v)_v. \quad (39)$$

Since

$$\gamma_v = \gamma^3 v c^{-2}, \quad (40)$$

equation (39) becomes

$$\gamma = w (\gamma^2 + \gamma^4 v^2 c^{-2}), \quad (41)$$

or

$$\gamma = w \gamma^2 (1 + \gamma^2 v^2 c^{-2}) = w \gamma^4 \quad (42)$$

which is obviously false. This contradiction shows that the equations (36) and (19) are not satisfied. Therefore, the relativistic matrix B^2 (8) is not Hamiltonian except when $P = 0$.

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